1. Let $n = 4k^2$. Consider the following Boolean function $f : \{0,1\}^n \to \{0,1\}$. View the n variables $\mathbf{x}$ as a $2k \times 2k$ matrix $\mathbf{x} = \{x_{i,j}\}$, $i, j \in [2k]$. Then $f(\mathbf{x}) = 1$ if there exists a row of the form $0^a 1 1 0^b$, i.e., there are two consecutive 1's and all other variables are 0's. What is exactly $D(f)$? (Note that you need to give both a query algorithm for the upper bound part and an adversary strategy for the lower bound part.)

2. Let NAE denote the following function: $\mathsf{NAE}(x_1, x_2, x_3) = 1$ if the three bits x_1, x_2, x_3 are not all equal; $\mathsf{NAE}(x_1, x_2, x_3) = 0$ if $x_1 = x_2 = x_3 = 0$ or 1. Let $n = 3^k$ and $f : \{0,1\}^n \to \{0,1\}$ denote the Boolean function where we apply k levels of NAE over the 3^k input variables. What are $s(f)$ and $D(f)$?

3. A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is monotone if for all $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ such that $\mathbf{x} \le \mathbf{y}$, i.e., $x_i \le y_i$ for all $i \in [n]$, we have $f(\mathbf{x}) \le f(\mathbf{y})$. Show that $s(f) = \mathrm{bs}(f) = C(f)$ for all monotone Boolean functions. Conclude that $D(f) \le \mathrm{bs}(f)^2$ and this quadratic gap is tight for monotone Boolean functions.

4. Show that every function $f : \{0,1\}^n \to \mathbb{R}$ can be uniquely expressed as a multilinear polynomial p:

$$p(\mathbf{x}) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i, \quad \text{where each } c_S \in \mathbb{R},$$

such that $p(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in \{0,1\}^n$.

5. Use Markov's inequality to prove the following corollary we used in class. If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial that satisfies $c \le p(x) \le d$ for all $a \le x \le b$, then we have

$$\max_{a \le x \le b} \left| p'(x) \right| \le \left(\mathsf{deg}(p) \right)^2 \cdot \frac{d - c}{b - a}.$$

6. Show that if $f : \{0,1\}^n \to \{0,1\}$ is non-constant, symmetric, and monotone, then $\mathsf{deg}(f) = n$. (Hint: Symmetrization, and the connection between the number of roots and degree of a univariate polynomial.)