

1. Let $f : \{\pm 1\}^n \rightarrow \{\pm 1\}$ and T be a decision tree that computes f . For any $\mathbf{x} \in \{\pm 1\}^n$ and $i \in [n]$, let

$$\text{sign}_i(\mathbf{x}) = \begin{cases} x_i & \text{if the } i\text{th variable is queried by } T \text{ upon } \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$$

Show that $\mathbb{E}_{\mathbf{x}}[\text{sign}_i(\mathbf{x})] = 0$ and $\mathbb{E}_{\mathbf{x}}[\text{sign}_i(\mathbf{x}) \cdot \text{sign}_j(\mathbf{x})] = 0$ for all $i \neq j \in [n]$ when \mathbf{x} is drawn uniformly at random from $\{\pm 1\}^n$.

2. (Rank Lower Bound) Let $f : X \times Y \rightarrow \{0, 1\}$. Let M_f denote the following $|X| \times |Y|$ matrix: the rows are indexed by elements $x \in X$; the columns are indexed by elements $y \in Y$; the (x, y) th entry is $f(x, y)$. Let $r = \text{rank}(M_f)$ denote the rank of M_f over the field of reals. **1)** Show that $D(f) \geq \log_2 r$, where $D(f)$ denotes the deterministic communication complexity of f (same below). Can you strengthen it to

$$D(f) \geq \log_2(2r - 1)$$

2) Show that $D(f) \leq r + 1$. Closing this exponential gap remains an important open problem. Currently the best upper bound is $D(f) = O(r^{1/2} \log r)$.

3. Given an undirected graph G over n vertices, we define the following “clique vs. independent set” problem with respect to G : Alice receives as an input C , which is a clique in G (a set of vertices with an edge between any two of them). Bob receives as an input I , which is an independent set in G (a set of vertices with no edges between them). The function $\text{CIS}_G(C, I)$ is defined as the size of $C \cap I$ (observe that this size is either 0 or 1). Prove that for any G , $\Omega(\log n) \leq D(\text{CIS}_G) \leq O(\log^2 n)$. Moreover, if there is a $c < 2$ such that $D(\text{CIS}_G) \leq O(\log^c n)$ for all G , then show that for any $f : X \times Y \rightarrow \{0, 1\}$,

$$D(f) = O\left(\left(\log C^D(f)\right)^c\right),$$

where $C^D(f)$ denotes the *partition number* of f : the smallest number of monochromatic rectangles needed in a (pairwise disjoint) partition of $X \times Y$. (In class we used the fact that $D(f) \geq \log C^D(f)$.)

4. (Communication Complexity of Relations) A *legal* relation R is a subset $R \subseteq X \times Y \times Z$ such that for any $x \in X$ and $y \in Y$ there exists at least one $z \in Z$ such that $(x, y, z) \in R$. The communication problem that R defines is the following: Alice is given $x \in X$, Bob is given $y \in Y$, and their task is to find *some* $z \in Z$ such that $(x, y, z) \in R$. We say a deterministic protocol computes R if for every (x, y) , the leaf it reaches is labeled an element $z \in Z$ such that $(x, y, z) \in R$. (Note that each leaf is labeled a value $z \in Z$.) We use $D(R)$ to denote the cost (depth) of the best deterministic protocol that computes R .

Let $X \subset \{0, 1\}^n$ denote the set of all strings whose parity is 1 (that is, \mathbf{x} such that $\sum_{i=1}^n x_i$ is odd) and $Y \subset \{0, 1\}^n$ denote the set of all strings whose parity is 0. Let R denote the following relation: $(\mathbf{x}, \mathbf{y}, i) \in R$ if $\mathbf{x} \in X$, $\mathbf{y} \in Y$ and $x_i \neq y_i$. Show that $D(R) = O(\log n)$ (this is actually tight and the lower bound can be proved by considering the partition number of R).

5. (Karchmer and Wigderson Game) Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. We are interested in the minimum depth of all Boolean circuits that compute f , denoted by $d(f)$ (google to find the definition of Boolean circuits.) Here we only consider circuits in which every gate is either AND or OR and takes two inputs; every input node is labeled by either a variable x_i or a negated variable \bar{x}_i . There is an interesting connection between $d(f)$ and the communication complexity of a relation defined using f as follows. Let $X = f^{-1}(1)$ denote the set of $\mathbf{x} \in \{0, 1\}^n$ such that $f(\mathbf{x}) = 1$ and $Y = f^{-1}(0)$. Let $R_f \subseteq X \times Y \times [n]$ consist of all triples $(\mathbf{x}, \mathbf{y}, i)$ such that $x_i \neq y_i$. Show that $d(f) = D(R_f)$.

6. (Distributional Complexity) Let μ be a probability distribution over $X \times Y$. The (μ, ϵ) -distributional complexity of $f : X \times Y \rightarrow \{0, 1\}$, denoted by $D_\epsilon^\mu(f)$, is the minimum cost (depth) of all deterministic protocols that give the correct answer for f with probability at least $1 - \epsilon$, when the input pair (x, y) is sampled from μ . Show that $R_\epsilon^{\text{pub}}(f) = \max_\mu D_\epsilon^\mu(f)$. (Here we use $R_\epsilon^{\text{pub}}(f)$ to denote the minimum cost (depth) of all public-coin randomized protocols for f that give the correct answer for every pair (x, y) with probability at least $1 - \epsilon$.)