1. Let \( f : \{\pm 1\}^n \rightarrow \{\pm 1\} \) and \( T \) be a decision tree that computes \( f \). For any \( x \in \{\pm 1\}^n \) and \( i \in [n] \), let

\[
\text{sign}_i(x) = \begin{cases} 
  x_i & \text{if the } i\text{th variable is queried by } T \text{ upon } x \\
  0 & \text{otherwise}
\end{cases}
\]

Show that \( E_x[\text{sign}_i(x)] = 0 \) and \( E_x[\text{sign}_i(x) \cdot \text{sign}_j(x)] = 0 \) for all \( i \neq j \in [n] \) when \( x \) is drawn uniformly at random from \( \{\pm 1\}^n \).

2. (Rank Lower Bound) Let \( f : X \times Y \rightarrow \{0, 1\} \). Let \( M_f \) denote the following \(|X| \times |Y|\) matrix: the rows are indexed by elements \( x \in X \); the columns are indexed by elements \( y \in Y \); the \((x, y)\)th entry is \( f(x, y) \). Let \( r = \text{rank}(M_f) \) denote the rank of \( M_f \) over the field of reals. 1) Show that \( D(f) \geq \log_2 r \), where \( D(f) \) denotes the deterministic communication complexity of \( f \) (same below). Can you strengthen it to

\[
D(f) \geq \log_2 (2r - 1)
\]

2) Show that \( D(f) \leq r + 1 \). Closing this exponential gap remains an important open problem. Currently the best upper bound is \( D(f) = O(r^{1/2} \log r) \).

3. Given an undirected graph \( G \) over \( n \) vertices, we define the following “clique vs. independent set” problem with respect to \( G \): Alice receives as an input \( C \), which is a clique in \( G \) (a set of vertices with an edge between any two of them). Bob receives as an input \( I \), which is an independent set in \( G \) (a set of vertices with no edges between them). The function \( \text{CIS}(C, I) \) is defined as the size of \( C \cap I \) (observe that this size is either 0 or 1). Prove that for any \( G \), \( \Omega(\log n) \leq D(\text{CIS}(G)) \leq O(\log^2 n) \). Moreover, if there is a \( c < 2 \) such that \( D(\text{CIS}(G)) \leq O(\log^c n) \) for all \( G \), then show that for any \( f : X \times Y \rightarrow \{0, 1\} \),

\[
D(f) = O\left( (\log C^D(f))^{\frac{c}{2}} \right),
\]

where \( C^D(f) \) denotes the partition number of \( f \): the smallest number of monochromatic rectangles needed in a (pairwise disjoint) partition of \( X \times Y \). (In class we used the fact that \( D(f) \geq \log C^D(f) \).

4. (Communication Complexity of Relations) A legal relation \( R \) is a subset \( R \subseteq X \times Y \) such that for any \( x \in X \) and \( y \in Y \) there exists at least one \( z \in Z \) such that \((x, y, z) \in R\). The communication problem that \( R \) defines is the following: Alice is given \( x \in X \), Bob is given \( y \in Y \), and their task is to find some \( z \in Z \) such that \((x, y, z) \in R\). We say a deterministic protocol computes \( R \) if for every \((x, y)\), the leaf it reaches is labeled an element \( z \in Z \) such that \((x, y, z) \in R\). (Note that each leaf is labeled a value \( z \in Z \).)

We use \( D(R) \) to denote the cost (depth) of the best deterministic protocol that computes \( R \).

Let \( X \subset \{0, 1\}^n \) denote the set of all strings whose parity is 1 (that is, \( x \) such that \( \sum_{i=1}^n x_i \) is odd) and \( Y \subset \{0, 1\}^n \) denote the set of all strings whose parity is 0. Let \( R \) denote the following relation: \((x, y, i) \in R \) if \( x \in X \), \( y \in Y \) and \( x_i \neq y_i \). Show that \( D(R) = O(\log n) \) (this is actually tight and the lower bound can be proved by considering the partition number of \( R \)).
5. (Karchmer and Wigderson Game) Let \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) be a Boolean function. We are interested in the minimum depth of all Boolean circuits that compute \( f \), denoted by \( d(f) \) (google to find the definition of Boolean circuits.) Here we only consider circuits in which every gate is either AND or OR and takes two inputs; every input node is labeled by either a variable \( x_i \) or a negated variable \( \bar{x}_i \). There is an interesting connection between \( d(f) \) and the communication complexity of a relation defined using \( f \) as follows. Let \( X = f^{-1}(1) \) denote the set of \( x \in \{0, 1\}^n \) such that \( f(x) = 1 \) and \( Y = f^{-1}(0) \). Let \( R_f \subseteq X \times Y \times [n] \) consist of all triples \((x, y, i)\) such that \( x_i \neq y_i \). Show that \( d(f) = D(R_f) \).

6. (Distributional Complexity) Let \( \mu \) be a probability distribution over \( X \times Y \). The \((\mu, \epsilon)\)-distributional complexity of \( f : X \times Y \rightarrow \{0, 1\} \), denoted by \( D^\mu_\epsilon(f) \), is the minimum cost (depth) of all deterministic protocols that give the correct answer for \( f \) with probability at least \( 1 - \epsilon \), when the input pair \((x, y)\) is sampled from \( \mu \). Show that \( R^\text{pub}_\epsilon(f) = \max_\mu D^\mu_\epsilon(f) \). (Here we use \( R^\text{pub}_\epsilon(f) \) to denote the minimum cost (depth) of all public-coin randomized protocols for \( f \) that give the correct answer for every pair \((x, y)\) with probability at least \( 1 - \epsilon \).)