1 Deterministic complexity measures and their relations

In the previous lecture we have seen that it holds that \( D(f) \geq C(f) \geq bs(f) \geq s(f) \) for boolean functions \( f : \{0, 1\}^n \to \{0, 1\} \). It is an open question if \( D(f) \) can be upper bounded by a power of \( s(f) \), e.g. \( (s(f))^{100} \geq D(f) \geq s(f) \).

Example 1. Let \( n = 4k^2 \) and let \( f \) be a function that takes as input a boolean matrix of dimensions \( 2k \times 2k \). The function outputs 1 if there is a row of the form \( 0^a110^b \) for \( a, b \geq 0 \), otherwise it outputs 0.

For the above function we have that:

- \( s_x(f) = \sqrt{n} \) for the worst case input \( x \) with one row congaing exactly two consecutive 1 that allows you many ways to change the value by flipping a bit.
- \( bs(f) = n/2 \), for the zero matrix input were we need to flip two consecutive bits that count as a block to 1 to change the function's output.
- \( D(f) = n \).

The above example is the biggest gap we have between the \( s(f) \), \( bs(f) \) measures. It is an open question whether there are functions \( f \) for which \( bs(f) \gg (s(f))^2 \). Another open questions is whether there is a constant \( k \) such that \( bs(f) \leq O(s^k(f)) \) for all functions \( f \).

Theorem 2. It holds that \( D(f) \leq (bs(f))^3 \).

Proof. We need to show the following:

I. \( C(f) \leq s(f) \cdot bs(f) \)

II. \( D(f) \leq C_1(f) \cdot bs(f) \), \( D(f) \leq C_0(f) \)

Lemma 3. If \( B \subseteq [n] \) is a minimal sensitive block for \( x \) then \( |B| \leq s(f) \)

Proof. As \( B \) is a minimal sensitive block for \( x \) we have that \( f(x) \neq f(x^{(B)}) \). If we pick an \( i \in B \) then it must be the case that \( f(x) = f(x^{(B-\{i\})}) \neq f(x^{(B)}) \) since \( B \) is minimal. Thus we have \( s(f) \geq s_{x(B)}(f) \geq |B| \).

Proof. (I.) : Given \( x, f \) we need to find a certificate \( C \) for \( x \) with \( |C| \leq s(f) \cdot bs(f) \). Let \( B_1, \ldots, B_b \) be disjoint minimal sensitive blocks for \( x \) with \( b = bs_x(f) \). Thus we have that \( |B_i| \leq s(f) \). Our goal is to show that \( C = \cup_i B_i \) is a certificate for \( x \). If it wasn’t there would be a a \( y \) such that \( y/c = x/c \) but \( f(y) \neq f(x) \). Thus there is a \( B \) such that \( B \cap C = \emptyset \) and by having \( y = x^{(B)} \) we contradict the fact that \( b = bs_x(f) \).

\( \Box \)
Proof. (II.) : (The proof is for $C_1(f)$, similarly for $C_0(f)$). For this part we design an algorithm $A$ deciding $f(x)$:

- Maintain a set $X \subseteq \{0,1\}^n$ which contains all strings that are consistent with the query results so far (initially $X = \{0,1\}^n$).
- Repeat $bs(f)$ many times:
  - Check is all $x \in X$ have the same $f(x) = b$. If yes then output $b$.
  - If there is a $y \in X$ with $f(y) = 1$:
    * We can find a certificate $C$ of $y$ with $|C| = C_y(f)$. If there is no such $C$ we return 0.
    * We query all variables in $C$ that have not yet been queried.
    * We update $X$.
- We check if all $x \in X$ have the same $f(x)$.

The above algorithm terminates with $bs(f)$ loops and queries $C_1(f)$ variables in each iteration. For correctness proof we assume towards contradiction that during the check step that there is a $z \in \{0,1\}^n$ such that $f(z) = 0$ that survived. In loop 1 we picked a $y_1$ that is consistent with all queries so far and all variables in $C_1$ are queried.

**Claim 4.** There is a $B_1 \subseteq C_1$ such that $B_1$ is sensitive block for $z$.

Then in loop 2 we pick a $y_2$ that is consistent with queries so far and $C_2$ is a certificate for $y_2$ (with $f(y_2) = 1$). Thus all variables in $C_2 - C_1$ are queried.

**Claim 5.** There is a $B_2 \subseteq C_2 - C_1$ such that $B_2$ is a sensitive block for $z$.

$C_2$ will have some bits same as $C_1$ along with new bits. We know that $z/C_2 \neq y_2/C_2$ and $z/C_1 \neq y_2/C_1$. By the end of the algorithm we have $b+1$ disjoint sensitive blocks $B_1, \ldots, B_{b+1}$ ($B_i \subseteq C_i - C_{i-1}$) for $z$ contradicting the block sensitivity being $bs(f)$.

2 The degree of a function

**Definition 6.** We define the degree of $f : \{0,1\}^n \to \{0,1\}$ $\deg(f)$ as the degree of the polynomial $p : \mathbb{R}^n \to \mathbb{R}$ such that $p(x) = f(x)$ for all $x \in \{0,1\}^n$.

We observe that the above polynomial will be a multilinear polynomial, i.e. a polynomial of the form $p(x) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i$.

**Lemma 7.** For every $f$ there is a unique multilinear polynomial $p$ such that $p(x) = f(x)$ over all $x \in \{0,1\}^n$.

The degree of $f$ $\deg(f)$ will be the degree of this unique $p$, so it is well defined.
Example 8. Let \( n = 3^k \) and consider the function consisting of \( k \) levels of the function \( E(x_1x_2x_3) \) which is 1 there are either 1 or 2 “1” in \( (x_1,x_2,x_3) \), otherwise it is 0.

We have that \( \deg(f) = 2^k \) and the corresponding polynomial is \( p_E = x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 \) over all \( x \in \{0,1\}^3 \). It also holds that \( D(f) = 3^k \).

Theorem 9. It holds that \( D(f) \geq \deg(f) \).

Proof. We can see that from constructively by following a path along the decision tree of \( f \). For example the path \((x_1)\) with value \( 0 \rightarrow (x_2) \) with value \( 1 \rightarrow (x_3) \) with value \( 0 \rightarrow 1 \), from the root of the decision tree to a leaf with value 1 can be represented as \((1 - x_1)x_2(1 - x_3)\). Thus the degree of the polynomial representing each path is at most \( D(f) \).

Theorem 10. It holds that \( D(f) \leq O(\deg^4(f)) \).

To show that we need to prove that:

A. \( bs(f) \leq 2(\deg(f))^2 \)

B. \( D(f) \leq (\deg(f))^2 \cdot bs(f) \leq (\deg(f))^4 \).

Lemma 11. Markov brother’s inequality: For every polynomial \( p : \mathbb{R} \to \mathbb{R} \) of degree \( d \) it holds that

\[
\max_{x \in [-1,1]} |p'(x)| \leq d^2 \max_{x \in [-1,1]} |p(x)|.
\]

Corollary 12. If for a polynomial \( p \) of degree \( d \) we have \( c \leq p(x) \leq d \) for \( a \leq x \leq b \) then

\[
\max_{x \in [a,b]} |p'(x)| \leq d^2 \frac{d-c}{b-a}.
\]

Theorem 13. For a polynomial \( p : \mathbb{R} \to \mathbb{R} \) with \( b_1 \leq p(i) \leq b_2 \) for all integers \( 0 \leq i \leq n \) and \( |p'(x)| \geq c \) for some \( 0 \leq x \leq n \) it holds that \( \deg(p) \geq \sqrt{cn/(c + b_2 - b_1)} \).

Proof. Let \( \beta = \max\{0, \max_{0 \leq x \leq n} p(x) - b_2, \max_{0 \leq x \leq n} b_1 - p(x)\} \).

- Case \( \beta = 0 \): Then we apply the corollary for \( a = 0, b = n, c = b_1, d = b_2 \) so that we have

  \[
  (\deg(p))^2 \frac{b_2 - b_1}{n} \geq c \Rightarrow \deg(p) \geq \sqrt{\frac{nc}{b_2 - b_1}}.
  \]

- Case \( \beta > 0, b = \max_{0 \leq x \leq n} \). Then we apply the corollary for \( a = 0, b = n, c = b_1 - \beta, d = b_2 + \beta \) so that we have

  \[
  (\deg(p))^2 \frac{b_2 - b_1 + 2\beta}{n} \geq c.
  \]

  In this case there must be a \( X \) such that \( |p'(X)| \geq 2\beta \) and we can use \( 2\beta \) in the place of \( c \), having that \( (\deg(p))^2 \frac{b_2 - b_1 + 2\beta}{n} \geq 2\beta \). From the above we conclude that

  \[
  \frac{(\deg(p))^2}{n} \geq \max\{ \frac{c}{b_2 - b_1 + 2\beta}, \frac{2\beta}{b_2 - b_1 + 2\beta} \}
  \]

To prove theorem 10 [A.] we need to prove that \( \deg(f) \geq \sqrt{\frac{3}{2}} \) with \( b = bs(f) \). The full proof is given in the following lecture. For now we describe the steps towards that.

First we consider a polynomial \( p \) of degree \( \deg(f) = d \) such that \( p(x) = f(x) \) for all \( x \in \{0,1\}^n \).

- We construct a new polynomial \( g(y_1 \ldots y_k) \) from \( p \) with: \( \deg(g) \leq \deg(f) \), \( g(\overline{0}) = 0 \) and \( g(\overline{1}) = 1 \).
• We construct a symmetric function $h$ from $g$ (due to Minsky-Papert), being $h(x)\sum_{\pi \in S_b} \frac{g(x_{\pi})}{b!}$. A symmetric function is a function that its outcome does not change if we permute the input variables.

  – The function $h$ is symmetric as its value depends only on the Hamming weight of the input. That is there is a function $f^* : [0, n] \rightarrow \mathbb{R}$ such that $h(x) = h^*(|x|)$.
  – $h^*$ has degree $\leq \deg(g)(\leq \deg(f))$ and $h^*(0) = 0, h^*(1) = 1$.
  – We can now use the corollary on $h^*$